(Fun) Facts about (Continuous) Fourier Series

Basic Formulas for $f(t)$ periodic with period 2π

Sine/Cosine Series (from MAT 219)

$$f(t) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt)$$

$$\to \left\{ \begin{aligned} a_n &= 1/\pi \int_{-\pi}^{\pi} f(t) \cdot \cos(nt)\, dt \\ b_n &= 1/\pi \int_{-\pi}^{\pi} f(t) \cdot \sin(nt)\, dt \end{aligned} \right\}$$

Complex Exponential Series

$$f(t) = \sum_{-\infty}^{\infty} c_n e^{int} \qquad (c_{-n} = \overline{c_n})$$

$$\to \left\{ c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \cdot e^{-int}\, dt \right.$$

If $f(t)$ has period $2L$ then

$\to$ { change all π to L; change all t to $(\pi/L\, t)$

Comparison of series:

$$f(t) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt)$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{int} + \overline{c_n} e^{-int}$$

$(\overline{c_n} = c_{-n})$

"frequency 0 component" = average value of $f(t)$

"frequency n component"

$\to a_0/2 = c_0$ (= average value of $f(t)$)

$\to a_n \cos(nt) + b_n \sin(nt) = c_n e^{int} + \overline{c_n} e^{-int}$

[Recall: $e^{int} = \cos(nt) + i \sin(nt)$]

Conversion:

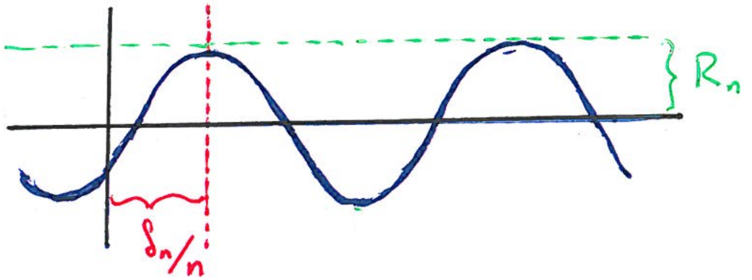
$$\left\{ \begin{aligned} c_n &= \frac{a_n - i b_n}{2} \\ c_{-n} &= \frac{a_n + i b_n}{2} \end{aligned} \right\} \qquad \left\{ \begin{aligned} a_n &= 2\,\mathrm{Re}(c_n) \\ b_n &= -2\,\mathrm{Im}(c_n) \end{aligned} \right\}$$

WARNING: Highlighted parts are easy to forget!

Properties:

Recall: $a_n\cos(nt) + b_n\sin(nt) = R_n\cos(nt - \delta_n)$

$$\begin{cases} R_n \text{ is amplitude} & R_n^2 = a_n^2 + b_n^2 \\ \delta_n \text{ is phase} & \tan(\delta_n) = b_n/a_n \end{cases}$$



$$\left\{\begin{matrix}\text{Amplitude of frequency} \\ n \text{ component of } f(t)\end{matrix}\right\} = \sqrt{a_n^2 + b_n^2}$$

$$= 2|c_n|$$

length(c_n)

$$\left\{\begin{matrix}\text{Phase of frequency} \\ n \text{ component of } f(t)\end{matrix}\right\} = \arctan(b_n/a_n)$$

$$= -\mathrm{Arg}(c_n)$$

Argument (c_n)

$$\Longrightarrow \underline{c_n = \frac{R_n}{2}\cdot e^{-i\delta_n}}$$

$$\{\text{Energy of } f(t)\} = \int_{-\pi}^{\pi} (f(t))^2\,dt$$

$$= 2\pi a_0^2 + \pi\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

$$= 2\pi\sum_{n=0}^{\infty}|c_n|^2$$

(square of length)

Basic Manipulations:

(Similar to Laplace transforms $\mathcal{L}\{f(t)\}$ in 219)

Notation: $\mathcal{F}\{f(t)\}$ = Fourier coefficients of $f(t)$

$= (c_0, c_1, c_2, \dots)$

(**Recall** $c_{-1} = \overline{c_1}$, $c_{-2} = \overline{c_2}$ etc)

$\mathcal{F}_n\{f(t)\} = n^{th}$ Fourier coefficient

$= c_n$

$$\mathcal{F}_n\{f(t)\} = c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\cdot e^{-int}\,dt$$

"linearity"

$$\mathcal{F}_n\{f(t) + g(t)\} = \mathcal{F}_n\{f(t)\} + \mathcal{F}_n\{g(t)\}$$

$$\mathcal{F}_n\{k\cdot f(t)\} = k\,\mathcal{F}_n\{f(t)\}$$

Manipulations (Continued):

$$\mathcal{F}_n\{f(t-s)\} = e^{-ins}\cdot \mathcal{F}_n\{f(t)\}$$

$$\mathcal{F}_n\left\{\frac{d}{dt} f(t)\right\} = in \cdot \mathcal{F}_n\{f(t)\} \quad (** \text{ see p4})$$

Note: $\mathcal{F}_0\left\{\frac{d}{dt} f(t)\right\} = 0$

If $\mathcal{F}_0\{f(t)\} = 0$, then

$$\mathcal{F}_n\left\{\int f(t)\,dt\right\} = -\frac{i}{n}\cdot \mathcal{F}_n\{f(t)\}$$

(If $\mathcal{F}_0\{f(t)\} \neq 0$, then (*)

$$\mathcal{F}_n\left\{\int f(t)\,dt\right\} = -\frac{i}{n}\mathcal{F}_n\{f(t)\} + \underbrace{\frac{(-1)^n}{n} i}_{\mathcal{F}_n\{t\}}\, \mathcal{F}_0\{f\}$$
)

Basic Values:

- $e^{in\pi} = (-1)^n$
- $e^{-in\pi} = (-1)^n$
- $e^{in\pi/2} = (i)^n$
- $e^{-in\pi/2} = (-1)^n (i)^n$

Note on Even/Odd Cosine/Sine Real/Imaginary

$f(t)$ $\boxed{\text{even}}$ $\Longleftrightarrow$ Cosine/Sine series has only $\boxed{\cos(nt)}$ $\Longleftrightarrow$ Exponential Series is $\boxed{\text{Real}}$

$f(-t) = f(t)$ $\qquad b_n = 0$ $\qquad \mathrm{Im}(c_n) = 0$

$f(t)$ $\boxed{\text{odd}}$ $\Longleftrightarrow$ Cosine/Sine series has only $\boxed{\sin(nt)}$ $\Longleftrightarrow$ Exponential Series is $\boxed{\text{Imaginary}}$

$f(-t) = -f(t)$ $\qquad a_n = 0$ $\qquad \mathrm{Re}(c_n) = 0$

Basic Fourier Coefficient Examples:

- Fourier coefficients of $f(t) = 1$

  $\mathcal{F}_0\{1\} = 1$ other $\mathcal{F}_n\{1\} = 0$

- Fourier coefficients of $f(t) = t$

  $\mathcal{F}_0\{t\} = 0$

  $\mathcal{F}_n\{t\} = (-1)^n \frac{1}{n}\cdot i$ (for $n \neq 0$)

  → see (*)

- Fourier coefficients of $f(t) = \delta(t)$

  $\mathcal{F}_n\{\delta(t)\} = \frac{1}{2\pi}$

(**) About $\frac{d}{dt}$ and $\mathcal{F}_n$ ...

The derivative rule

$$\mathcal{F}_n\left\{\frac{d}{dt} f(t)\right\} = in\, \mathcal{F}_n\{f(t)\}$$

is not always true.

Note that $in\,\mathcal{F}_n\{f(t)\}$ is a sequence (in n)
the derivative rule only works when

$$\boxed{\lim_{n\to\infty} in\,\mathcal{F}_n\{f(t)\} = 0}$$

For most functions this will be true
→ but not all!

EX: $\begin{cases} \mathcal{F}_n\{t\} = (-1)^n \frac{1}{n} \cdot i & (n \neq 0) \\ \mathcal{F}_0\{t\} = 0 \end{cases}$

$$1 = \frac{d}{dt} t$$

$$\mathcal{F}_n\{1\} \neq in\, \mathcal{F}_n\{t\} = in \cdot (-1)^n \frac{1}{n} \cdot i = (-1)^{n+1}$$

$\lim \neq 0.$

(Actually $\mathcal{F}_0\{1\} = 1$ other $\mathcal{F}_n\{1\} = 0$.)

More Basic Fourier Coefficient Examples:

- Fourier coefficients of $f(t) = u_0(t)$

$$\mathcal{F}_0\{u_0(t)\} = \frac{1}{2}$$

$$\mathcal{F}_n\{u_0(t)\} = \frac{1}{2\pi n}\left((-1)^n - 1\right) i$$

- Fourier coefficients of $f(t) = \begin{cases} 1 & 0<t<h \\ 0 & \text{otherwise} \end{cases}$

(f = "pulse" )

$$\mathcal{F}_0\{\text{pulse}\} = \frac{h}{2\pi}$$

$$\mathcal{F}_n\{\text{pulse}\} = \frac{1}{2\pi n}\left(e^{-inh} - 1\right) i$$

More Manipulations:

Reflect across y-axis

$$\mathcal{F}_n\{f(-t)\} = \overline{\mathcal{F}_n\{f(t)\}}$$

conjugate